Topology and topological defects appearing in Bose-Einstein condensate

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Model Equations in Bose-Einstein Condensation and Related topics (Dec. 7, 2010)
Contents

1. Scalar BEC and quantized vortices
2. Topology and topological defects in spinor BEC
3. Summary
Atomic Bose-Einstein condensates

Dilute alkali atomic Bose-Einstein condensates has been realized in 1997

\[ ^{87}\text{Rb}, ^{23}\text{Na}, ^{7}\text{Li}, ^{1}\text{H}, ^{85}\text{Rb}, ^{41}\text{K}, ^{4}\text{He}, ^{133}\text{Cs}, ^{174}\text{Yb}, ^{52}\text{Cr}, ^{40}\text{Ca}, ^{84}\text{Sr} \]

- Trap of atoms
- Laser cooling of atoms
- Evaporating cooling of atoms
Hamiltonian and Gross-Pitaevskii equation

Mean-field Hamiltonian at the zero temperature

\[ \mathcal{H} = \int d\mathbf{x} \left[ \frac{\hbar^2}{2M} |\nabla \Psi|^2 + \frac{c_0}{2} |\Psi|^4 \right] \]

Time evolution of BEC: Gross-Pitaevskii equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta \mathcal{H}}{\delta \Psi} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi|^2 \right] \Psi \]

\[ \Psi = |\Psi| \exp[i\phi] \]

\[ \rho = |\Psi|^2 : \text{Density of BEC} \]

\[ \mathbf{v} = \frac{\hbar}{m} \nabla \phi : \text{Velocity field of BEC} \]
$U(1)$ gauge symmetry breaking in BEC

$$\mathcal{H} = \int d\mathbf{x} \left[ \frac{\hbar^2}{2M} |\nabla \Psi|^2 + \frac{c_0}{2} |\Psi|^4 \right]$$

Invariant under $U(1)$ gauge transformation: $\Psi \rightarrow e^{i\phi} \Psi$ : symmetry $G$

In Bose condensed system, phase $\phi$ is fixed to one value: Symmetry $H = 1$

Order-parameter manifold (degrees of freedom of $\Psi$)

$$G/H \simeq U(1) \simeq S^1$$
Topological defects (fundamental group)

Along the closed path, by how many times the path rotates the singular point of $\Psi$ (fundamental group $\pi_1$)
Topological defects in BEC (quantized vortex)

\[ G/H \cong U(1) \cong S^1 \]

\[ \pi_1(S^1) \cong \mathbb{Z} \text{ (quantized vortex)} \]

Quantized vortices \((\rho(x) = 0)\)
around which phase \(\phi(x)\)
changes by integer multiple of \(2\pi\): quantized vortices

\[ \Psi = |\Psi| \exp[i\phi] \]
\[ \rho = |\Psi|^2 : \text{Density of BEC} \]
\[ \mathbf{v} = \frac{\hbar}{m} \nabla \phi : \text{Velocity field of BEC} \]
Experimental observation of vortices

Vortex lattice and its formation in atomic BEC

Vortex lattice in $^{87}$Rb BEC

K. W. Madison et al. PRL 86, 4443 (2001)
Numerical simulation of vortex lattice formation

2D GP equation (in non-dimensional form)

\[(i - \gamma) \frac{\partial \Psi}{\partial t} = \left[ -\frac{\nabla^2}{2} - \mu + \frac{\omega^2 x^2}{2} + c_0 |\Psi|^2 + i\Omega_z \mathbf{x} \times \nabla \right] \Psi\]


- \(\gamma\): dissipation term
- \(\mu\): chemical potential
- \(\omega\): harmonic trap potential
- \(\Omega_z\): external rotation

\[t = 0: \text{Stationary state with } \Omega_z = 0\]
Numerical simulation of vortex lattice formation

3D GP equation (in non-dimensional form)

\[
(i - \gamma) \frac{\partial \Psi}{\partial t} = \left[ -\frac{\nabla^2}{2} - \mu + \frac{\omega^2 x^2}{2} + c_0 |\Psi|^2 + i \Omega_z (\mathbf{x} \times \nabla)_z \right] \Psi
\]

K. Kasamatsu et al. PRA 71, 063616 (2005)

Vortex as line defect
Reconnection of vortices

\[ i \frac{\partial \Psi}{\partial t} = \left[ -\frac{\nabla^2}{2} + c_0 |\Psi|^2 \right] \Psi \]

Neumann boundary condition
BEC with larger manifold: 2-component BEC

Different atom species: $^{87}$Rb - $^{41}$K
Mixture of Isotope: $^{87}$Rb - $^{85}$Rb
Different hyperfine state: $^{87}$Rb ($F = 1, m_F = -1$) - $^{87}$Rb ($F = 2, m_F = 1$)

$$\mathcal{H} = \int d\mathbf{x} \sum_{i=0}^{1} \left[ \frac{\hbar^2}{2M} |\nabla \Psi_i|^2 + \frac{c_0}{2} |\Psi_i|^4 + \frac{c_1}{2} |\Psi_i|^2 |\Psi_{1-i}|^2 \right]$$

$$i\hbar \frac{\partial \Psi_0}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi_0|^2 + c_1 |\Psi_1|^2 \right] \Psi_0$$

$$i\hbar \frac{\partial \Psi_1}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi_1|^2 + c_1 |\Psi_0|^2 \right] \Psi_1$$

$$G/H \cong S^1 \times S^1$$
Physics of topological excitation in Bose-Einstein condensates

BEC with larger manifold: 2-component BEC

\[ G/H \cong S^1 \times S^1 \]

\[ i\hbar \frac{\partial \Psi_0}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + V + c_0|\Psi_0|^2 + c_1|\Psi_1|^2 \right] \Psi_0 \]

\[ i\hbar \frac{\partial \Psi_1}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + V + c_0|\Psi_1|^2 + c_1|\Psi_0|^2 \right] \Psi_1 \]

V. Schweikhard et al. PRL 93, 210403 (2004)
Spinor BEC

BEC with spin degrees of freedom

Hyperfine coupling of electron end nuclear spin

\[(F = I + L + S)\]

\[S = 1/2\]

<table>
<thead>
<tr>
<th>(^{87}\text{Rb}, ; ^{23}\text{Na}, ; ^{7}\text{Li}, ; ^{41}\text{K})</th>
<th>(F=1, ; 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{85}\text{Rb})</td>
<td>(F=2, ; 3)</td>
</tr>
<tr>
<td>(^{133}\text{Cs})</td>
<td>(F=3, ; 4)</td>
</tr>
<tr>
<td>(^{52}\text{Cr})</td>
<td>(S=3, ; I=0)</td>
</tr>
</tbody>
</table>
Spinor BEC

BEC with spin degrees of freedom

\[ F = 2 \]

\[ \{ \begin{align*}
  m_F &= 2 \\
  m_F &= 1 \\
  m_F &= 0 \\
  m_F &= -1 \\
  m_F &= -2
\end{align*} \]

\[ F = 1 \]

\[ \{ \begin{align*}
  m_F &= 1 \\
  m_F &= 0 \\
  m_F &= -1
\end{align*} \]

\[ ^{87}\text{Rb} \ (I = 3/2) \]

Spin 1 : 3-component BEC

\[ \Psi = (\psi_1, \psi_0, \psi_{-1}) \]

Multicomponent BEC labeled by magnetic sublevel \( m_F \)

Field gradient

-1 0 1
Spinor BEC

Stern-Gerlach experiment

\[ F = 1 \]

\[ F = 2 \]


Rotation of Spin can be observed

Theory of Spinor BEC

Hamiltonian of spinor Bosons

\[
H = -\int d\mathbf{x} \frac{\hbar^2}{2M} \nabla \Psi_m^\dagger(\mathbf{x}) \nabla \Psi_m(\mathbf{x}) \\
+ \frac{1}{2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \Psi_{m_1}^\dagger(\mathbf{x}_1) \Psi_{m_2}^\dagger(\mathbf{x}_2) V_{m_1 m_2 m'_1 m'_2} (\mathbf{x}_1 - \mathbf{x}_2) \Psi_{m'_2}(\mathbf{x}_2) \Psi_{m'_1}(\mathbf{x}_1)
\]

Low energy contact interaction \((l = 0)\)

\[
V_{m_1 m_2 m'_1 m'_2} (\mathbf{x}_1 - \mathbf{x}_2) = \delta(\mathbf{x}_1 - \mathbf{x}_2) \sum_{F=0,2,4} g_F \sum_{m_1,m_2,m'_1,m'_2,M} O_{m_1 m_2}^{F,M} \left(O_{m'_1 m'_2}^{F,M}\right)^*
\]

Physics of topological excitation in Bose-Einstein condensates
Mean-field Hamiltonian (spin-1)

\[ H = \int dx \left[ \frac{\hbar^2}{2M} \sum_{m=-1}^{1} \nabla \Psi_m^* \nabla \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} F^2 \right] \]

Density \quad Spin

\[
F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
F_- = F_+^T, \quad F_x = \frac{F_+ + F_-}{2}, \quad F_y = \frac{F_+ - F_-}{2i}
\]

\[
\rho = \sum_{m=-1}^{1} |\Psi_m|^2
\]

\[
F = (\Psi_1^* \Psi_0^* \Psi_{-1}^*) (F_x, F_y, F_z)
\]

Gauge and spin rotation symmetry of wave function are broken

\[
\Psi' = e^{i\phi} e^{-in \cdot F_\alpha} \Psi \quad (G = U(1)_F \times SO(3)_\phi)
\]
Possible phase

\[ H = \int dx \left[ \frac{\hbar^2}{2M} \sum_{m=-1}^{1} \nabla \Psi_m^* \nabla \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} F^2 \right] \]

\[ c_1 > 0 : \text{polar} \ (^{23}\text{Na BEC}) \]

\[ e^{i\phi} e^{-in \cdot \hat{F} \alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ \mathbf{F} = 0 \]

\[ \frac{G}{H} \sim \frac{U(1)_\phi \times S_F^2}{(\mathbb{Z}_2)_{\phi+F}} \]

\[ c_1 < 0 : \text{Ferromagnetic} \ (^{87}\text{Rb BEC}) \]

\[ e^{i\phi} e^{-in \cdot \hat{F} \alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

\[ \mathbf{F} \neq 0 \]

\[ \frac{G}{H} \sim SO(3)_{\phi+F} \]
Gross-Pitaevskii equation

\[ i\hbar \frac{\partial \Psi_{\pm}}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 + V + c_0 \rho \right) \Psi_{\pm} + c_1 \left( \frac{1}{\sqrt{2}} F_\mp \Psi_0 \pm F_z \Psi_{\pm} \right) \]

\[ i\hbar \frac{\partial \Psi_0}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 + V + c_0 \rho \right) \Psi_0 + \frac{c_1}{\sqrt{2}} \left( F_+ \Psi_1 + F_- \Psi_{-1} \right) \]
Graphical image by the spherical Harmonics

\[ \sum_{m=-1}^{1} \Psi_m Y_{2,m} = Y_{1,1} + Y_{1,0} + Y_{1,-1} \]

\[ \cos \theta \]

Polar

Ferromagnetic

\[ -e^{i\varphi} \sin \theta \]
Topological defect in Ferromagnetic state

\[
\frac{G}{H} \approx \frac{U(1)_\phi \times SO(3)_F}{(U(1))_{\phi+F}} \\
\approx SO(3)_{\phi+F}
\]

Gauge vortex

Spin vortex

Continuous transformation

Physics of topological excitation in Bose-Einstein condensates
Topological defect in Ferromagnetic state

\[ \pi_1[SO(3)_{\phi+F}] \cong \mathbb{Z}_2 \]

Doubly winding state is no longer topological defect

Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding

\[(\Psi_1, (\Psi_1, \Psi_0, \Psi_{-1}) = (0, 0, e^{-2i\varphi})^{-2i\varphi}/2)\]

Adiabatic change of quadratic magnetic field
Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding

\[ i\hbar \frac{\partial \Psi_m}{\partial t} = \left(-\frac{\hbar^2}{2M} \nabla^2 + V + B\right) \Psi_m + \text{interaction} \]

\[ B = \begin{pmatrix}
B_z & B_q e^{i\varphi}/\sqrt{2} & 0 \\
B_q e^{-i\varphi}/\sqrt{2} & 0 & B_q e^{i\varphi}/\sqrt{2} \\
0 & B_q e^{-i\varphi}/\sqrt{2}/\sqrt{2} & -B_z
\end{pmatrix} \]

Physics of topological excitation in Bose-Einstein condensates
Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding

\[ i\hbar \frac{\partial \Psi_m}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 + V + B \right) \Psi_m + \text{interaction} \]

\[ B = \begin{pmatrix}
  B_z & B_q e^{i\phi} / \sqrt{2} & 0 \\
  B_q e^{-i\phi} / \sqrt{2} & 0 & B_q e^{i\phi} / \sqrt{2} \\
  0 & B_q e^{-i\phi} / \sqrt{2} / \sqrt{2} & -B_z 
\end{pmatrix} \]

Topological defect in polar state

\[ e^{i\phi} \cos \theta \]

\[ \frac{G}{H} \approx \frac{U(1)_\phi \times SO(3)_F}{SO(2)_F \times (\mathbb{Z}_2)_{\phi+F}} \]

\[ \approx \frac{U(1)_\phi \times S^2_F}{(\mathbb{Z}_2)_{\phi+F}} \]

\[ \pi_1[G/H] \approx \mathbb{Z} \]

\( \pi \) gauge transformation
Vortex ring in polar phase

Vortex ring

Usual ring

Alice ring
Spin-2 case

\[ H = \int d\mathbf{x} \left[ -\Psi^*_m \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \mathbf{F}^2 + \frac{c_2}{2} |A_{20}|^2 \right] \]

\[ A_{00}(\mathbf{x}) = 2\Psi_2(\mathbf{x})\Psi_{-2}(\mathbf{x}) - 2\Psi_1(\mathbf{x})\Psi_{-1}(\mathbf{x}) + \Psi_0(\mathbf{x})^2 \]

Singlet-pair amplitude
Spin-2 case

\[ H = \int d\mathbf{x} \left[ -\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \mathbf{F}^2 + \frac{c_2}{2} |A_{20}|^2 \right] \]

**Uniaxial Nematic:**
\[
\Psi_u = (0, 0, 1, 0, 0)^T
\]
\[
U(1)_\phi \times \frac{S_F^2}{(\mathbb{Z}_2)_F}
\]

**Biaxial Nematic:**
\[
\Psi_b = (1, 0, 0, 0, 1)^T / \sqrt{2}
\]
\[
\frac{U(1)_\phi \times SO(3)_F}{(D_4)_\phi+F}
\]

**Cyclic:**
\[
\Psi_c = (1, 0, 0, \sqrt{2}, 1)^T / \sqrt{3}
\]
\[
\frac{U(1)_\phi \times SO(3)_F}{(T)_{\phi+F}}
\]

**Ferromagnetic:**
\[
\Psi_f = (1, 0, 0, 0, 0)^T
\]
\[
\frac{SO(3)_{\phi+F}}{(\mathbb{Z}_2)_{\phi+F}}
\]

\[ c_2 = 4c_1 \]

Physics of topological excitation in Bose-Einstein condensates

\[ 87 \text{Rb} \]
Topological defect in cyclic state

Topological defects can be labeled by 12 rotations keeping tetrahedron invariant.

Non-Abelian topological defect!

$$\frac{U(1)_\phi \times SO(3)_F}{(T)_{\phi+F}}$$
Topological excitation in cyclic state

1/2-spin vortex
Topological excitation in cyclic state

1/3 vortex

Physics of topological excitation in Bose-Einstein condensates
\[
\begin{align*}
\frac{i\hbar}{\partial t} \Psi_2 &= -\frac{\hbar^2}{2M} \nabla^2 \Psi_2 + c_0 \rho \Psi_2 + c_1 \left( F_- \Psi_1 + 2F_z \Psi_2 \right) + c_2 A_{20} \Psi_{-2}^* \\
\frac{i\hbar}{\partial t} \Psi_1 &= -\frac{\hbar^2}{2M} \nabla^2 \Psi_1 + c_0 \rho \Psi_1 + c_1 \left( \frac{\sqrt{6}}{2} F_- \Psi_0 + F_+ \Psi_2 + F_z \Psi_1 \right) - c_2 A_{20} \Psi_{-1}^* \\
\frac{i\hbar}{\partial t} \Psi_0 &= -\frac{\hbar^2}{2M} \nabla^2 \Psi_0 + c_0 \rho \Psi_0 + \frac{\sqrt{6}}{2} c_1 \left( F_- \Psi_{-1} + F_+ \Psi_1 \right) + c_2 A_{20} \Psi_0^* \\
\frac{i\hbar}{\partial t} \Psi_{-1} &= -\frac{\hbar^2}{2M} \nabla^2 \Psi_{-1} + c_0 \rho \Psi_{-1} + c_1 \left( \frac{\sqrt{6}}{2} F_+ \Psi_0 + F_- \Psi_{-2} - F_z \Psi_{-1} \right) - c_2 A_{20} \Psi_{-1}^* \\
\frac{i\hbar}{\partial t} \Psi_{-2} &= -\frac{\hbar^2}{2M} \nabla^2 \Psi_{-2} + c_0 \rho \Psi_{-2} + c_1 \left( F_+ \Psi_{-1} - 2F_z \Psi_{-2} \right) + c_2 A_{20} \Psi_{-2}^*
\end{align*}
\]
Collision of vortices with non-commutative charge forms a new “rung” vortex connecting two vortices.

Collision dynamics of topological excitations

Abelian excitation

Non-Abelian excitation

Large ring

Unraveling of link

New excitation

Linked non-Abelian excitations cannot unravel because of the formation of the new excitation.
Topological charge of topological excitation

Topological invariant of excitations can be fixed by a closed path encircling the excitations.
Passing dynamics is possible for Abelian case
Rung $BA^{-1}$ is formed through the collision.
Collision of Vortex

\[ A = B \]

\[ AA^{-1} = 1 \]

Rung disappears for the same charge resulting reconnection.
Linked Vortex Rings

\[ AB = BA \]

Linked excitations with non-Abelian invariants never unravel.

Physics of topological excitation in Bose-Einstein condensates
Quantum turbulence

Vortices

2D plane of the phase
Non-Abelian quantum turbulence

Turbulent behavior is strongly affected by topology

M. Kobayashi, et al. in press
Summary

• In BEC, various kinds of topological excitations can be realized.
• Dynamics of topological excitations are affected by the order-parameter manifold and can dominate the nature of the system.
Topological charge of topological excitation

Topological charge of vortex can be fixed by a closed path encircling the vortex
Physics of topological excitation in Bose-Einstein condensates
Collision of Same Vortices

Energetically unfavorable

reconnection
Physics of topological excitation in Bose-Einstein condensates

Collision of Different Commutative Vortices

Energetically unfavorable

Passing
Physics of topological excitation in Bose-Einstein condensates
Physics of topological excitation in Bose-Einstein condensates

Linked Vortices

Non-commutative

Linked vortices cannot untangle
What is topological excitations?

Nematic liquid crystal

Crystal

Nematic

Liquid

Translational symmetry breaking

Rotational symmetry breaking
What is topological excitations?

Topological excitations in nematic liquid crystal

Topological excitation related to rotational symmetry breaking
What is topological excitations?

States with topological excitations cannot be continuously transformed to states without topological excitations.
What is topological excitations?

This is topological excitation in 2D system but not topological excitation in 3D system.

Characteristics of topological excitations strongly depend on the internal degrees of freedom (topology) of the system.

This is always topological excitation.
Observation of topological excitations in nematic liquid crystal

near surface

far from surface
Biaxial nematic liquid crystal
In XY-spin system, local spin (order parameter) can be expressed by a point in a circle

→ Order-parameter manifold
Topological excitations can be characterized by how many times the state rotates the circle along the closed path.
Heisenberg-spin

Order parameter can be expressed by a point in a sphere

Topological excitations can never be stabilized
What is topological excitations?

Excitations in symmetry broken systems via phase transitions

Liquid $\rightarrow$ Solid transition (spontaneous symmetry breaking)

- Free energy is invariant under translational and rotational transformations
- System is also invariant under transformations

- Free energy is invariant under translational and rotational transformations
- System is not invariant under transformations (symmetry breaking)

Physics of topological excitation in Bose-Einstein condensates
What is topological excitations?

**Liquid**
Atoms are little influenced by other atoms.

**Solid**
Positions and orientations of atoms are strongly affected by other atoms and fixed (spontaneous symmetry breaking).
What is topological excitations?

Topological excitations appear in symmetry broken systems

In crystal

- **dislocation**
  - Topological excitation related to translational symmetry breaking

- **disclination**
  - Topological excitation related to rotational symmetry breaking
$U(1)$ gauge symmetry breaking in BEC

Mean-field Hamiltonian at the zero temperature

$$H = \int d\mathbf{x} \left[ \frac{\hbar^2}{2M} \nabla \Psi^*(\mathbf{x}) \nabla \Psi(\mathbf{x}) + \frac{c_0}{2} |\Psi(\mathbf{x})|^4 \right]$$

$$\Psi(\mathbf{x}) = |\Psi(\mathbf{x})| \exp[i\varphi(\mathbf{x})]$$

$$\rho(\mathbf{x}) = |\Psi(\mathbf{x})|^2 : \text{Fluid density}$$

$$\mathbf{v}(\mathbf{x}) = \frac{\hbar}{m} \nabla \varphi(\mathbf{x}) : \text{Fluid velocity}$$
Point-like excitation

Physics of topological excitation in Bose-Einstein condensates
Point-like excitation

Polar phase

\[ \frac{G}{H} \simeq \frac{U(1)_{\varphi} \times S^2_F}{(\mathbb{Z}_2)_{\varphi+F}} \]

Point-like excitation cannot exist in Ferromagnetic phase

\[ \frac{G}{H} \simeq SO(3)_{\varphi+F} \]
$\pi_3$ excitation and Hopf mapping


\[
\pi_3 \left[ \frac{U(1)_G \times (S^2)_S}{(\mathbb{Z}_2)_{G+S}} \right] \cong (\mathbb{Z})_S
\]
$\pi_3$ excitation and Hopf mapping

$t = 0.00 \ T_L$

$B = 0$
Point-like excitation and 2D skyrmion


\[ \pi_2 \left[ \frac{U(1)_G \times (S^2)_S}{(\mathbb{Z}_2)_{G+S}} \right] \cong \pi_2[(S^2)_S] \cong (\mathbb{Z})_S \]
Two 2D skyrmions

Physics of topological excitation in Bose-Einstein condensates
Inversion of topological invariant
Physics of topological excitation in Bose-Einstein condensates

Vorton excitation

vorton
Spin-2 case

\[ H = \int d\mathbf{x} \left[ -\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \mathbf{F}^2 + \frac{c_2}{2} |A_{20}|^2 \right] \]

**Uniaxial Nematic:**
\( \Psi_U = (0, 0, 1, 0, 0)^T \)
\( U(1)_\varphi \times \frac{S_F^2}{(Z_2)_F} \)

**Biaxial Nematic:**
\( \Psi_B = (1, 0, 0, 0, 1)^T / \sqrt{2} \)
\( U(1)_\varphi \times \frac{SO(3)_F}{(D_4)_\varphi + F} \)

**Cyclic:**
\( \Psi_C = (1, 0, 0, \sqrt{2}, 1)^T / \sqrt{3} \)
\( \frac{U(1)_\varphi \times SO(3)_F}{(T)_\varphi + F} \)

**Ferromagnetic:**
\( \Psi_F = (1, 0, 0, 0, 0)^T \)
\( \frac{SO(3)_\varphi + F}{(Z_2)_\varphi + F} \)

\( c_2 = 4c_1 \)

87Rb

Physics of topological excitation in Bose-Einstein condensates
Nematic phase of spin-2

\[ H = \int d\mathbf{x} \left[ -\Psi^*_m \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} \rho^2 + \frac{c_1}{2} \mathbf{F}^2 + \frac{c_2}{2} |A_{20}|^2 \right] \]

Uniaxial Nematic:
\[ \Psi_U = (0, 0, 1, 0, 0)^T \]
\[ U(1)_\phi \times \frac{S^2_F}{(\mathbb{Z}_2)_F} \]

Biaxial Nematic:
\[ \Psi_B = (1, 0, 0, 0, 1)^T / \sqrt{2} \]
\[ U(1)_\phi \times SO(3)_F \]
\[ \left( D_4 \right)_{\phi+F} \]

Two states are degenerate via another continuous degree of freedom

New order-parameter manifold
\[ U(1)_\phi \times \frac{S^4_F}{(\mathbb{Z}_2)_F} \]
\[ (\mathbb{Z}_2)_\phi+F \]
Quasi-Nambu-Goldstone current

S. Uchino, et al., PRL in press

Decay of vortex in biaxial nematic phase

Emission of quasi-Nambu-Goldstone current
Cyclic State vs. Singlet-trio Condensed State

For $c_1 > 0$, $c_2 > 0$

Singlet-trio condensed state (only $U(1)$ is broken)

$$|\Psi\rangle = \left[ e^{i\varphi} \left( \frac{\sqrt{2} a_0 \left( a_0^\dagger a_{-2}^\dagger - 3 a_1^\dagger a_{-1}^\dagger - 6 a_2^\dagger a_{-2}^\dagger \right) + 3 \sqrt{3} (a_1^\dagger a_{-2}^\dagger + a_{-1}^\dagger a_{2}^\dagger)}{\sqrt{35}} \right) \right]^{N/3} |0\rangle$$

Transition occurs under $\sim 1 \mu G$

Cyclic state ($U(1) \times SO(3)$ is broken)

$$|\Psi\rangle = \left[ \sum_m \Psi_m a_m^\dagger \right]^N |0\rangle$$

M. Koashi, and M. Ueda. PRL 84, 1066 (2000)

Physics of topological excitation in Bose-Einstein condensates
For $c_1 > 0$, $c_2 < 0$

Singlet-pair condensed state (only $U(1)$ is broken)

$$|\Psi\rangle = \left[ e^{i\varphi} \left( \hat{a}_0^\dagger a_1 a_1^\dagger a_2 a_2^\dagger \right) \right]^{N/2} |0\rangle$$

Transition occurs under $\sim 1 \mu G$

Nematic state ($U(1) \times SO(3)$ is broken)

$$|\Psi\rangle = \left[ \sum_m \Psi_m a_m^\dagger \right]^N |0\rangle$$

$$\Psi = e^{i\varphi} e^{-i\hat{F} \cdot \alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} / \sqrt{2}$$
Topological defects (2nd homotopy group)

Along the closed sphere, by how many times the sphere covers the singular point of $\Psi$ (2nd homotopy group $\pi_2$)
Topological defects (3nd homotopy group)

Along the closed 3D sphere, by how many times the sphere covers the singular point of $\Psi$ (3nd homotopy group $\pi_3$)