非可換量子渦
Non-Abelian Vortex

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Mar. 28, 2009, 日本物理学会第64回年次大会
1. Non-Abelian Vortices are realized in the cyclic phase of spin-2 Bose-Einstein condensates
2. Non-Abelian character becomes remarkable in collision dynamics of two vortices
   I. We numerically show.
   II. We algebraically confirm.
Vortex in Bose-Einstein Condensates

Vortices appears as line defects when symmetry breaking happens

- Vortices are Abelian for single-component BEC
- We here consider vortices called “Non-Abelian”

G. P. Bewley et al.
Nature 441, 588 (2006)

K. W. Madison et al.
PRL 86, 4443 (2001)
Quantized Vortex and Topological Charge

Topological charge of a vortex can be considered how order parameter changes around the vortex core.

Single component BEC:

- Topological charge can be expressed by integer $n$: $n = \pi_1(G/H) = \mathbb{Z}$.

When topological charge can be expressed by non-commutative algebra (first homotopy group $\pi_1$ is non-Abelian), we define vortices as “non-Abelian vortices.”

Non-Abelian vortices are realized in the cyclic phase of spin-2 BEC.
Spin-2 Bose-Einstein Condensate

cyclic phase

\[ e^{i\phi} e^{-ie\cdot \hat{F}_\alpha} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix} \]

\[ \sum_{m=-2}^{2} \Psi_m Y_{2,m} \]

\[ Y_{2,2} + Y_{2,1} + Y_{2,0} + Y_{2,-1} + Y_{2,-2} \]

headless triad: axes can interchange each other by \(2\pi/3\) gauge transformation (different from triad of \(^3\)He-A)
Vortices in Spinor BEC

$S = 1$ Polar phase

\[ e^{i\phi} e^{-i e \cdot \mathbf{F} \alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

headless vector

Half quantized vortex: spin & gauge rotate by $\pi$ around vortex core

Topological charge can be expressed by half integer (Abelian vortex)

$\pi$ gauge transformation
Vortices in Spin-2 BEC

1/2-spin vortex: triad rotate by $\pi$ around three axes $e_x, e_y, e_z$
Vortices in Spin-2 BEC

1/3 vortex: triad rotate by $2\pi/3$ around four axis $e_1, e_2, e_3, e_4$ and $2\pi/3$ gauge transformation

\[ e_1 = (1, 1, 1)/\sqrt{3}, \quad e_2 = (1, -1, -1)/\sqrt{3} \]
\[ e_3 = (-1, 1, -1)/\sqrt{3}, \quad e_4 = (-1, -1, 1)/\sqrt{3} \]
Vortices in Spin-2 BEC

2/3 vortex: triad rotate by $4\pi/3$ around four axis $e_1, e_2, e_3, e_4$ and $4\pi/3$ gauge transformation

$e_1 = (1, 1, 1)/\sqrt{3}, \quad e_2 = (1, -1, -1)/\sqrt{3}$

$e_3 = (-1, 1, -1)/\sqrt{3}, \quad e_4 = (-1, -1, 1)/\sqrt{3}$
Topological Charge of Vortices is Non-Abelian

Topological charge can be expressed by the element of tetrahedral group
→ non-Abelian vortex ( : \( \pi_1(G/H) \) includes \( T \))

\[ e_x = (1, 0, 0) \quad e_y = (0, 1, 0) \quad e_z = (1, 0, 0) \]
Collision Dynamics of Vortices

“Non-Abelian” character becomes remarkable when two vortices collide with each other

→ Numerical simulation of the Gross-Pitaevskii equation

Initial state : two straight vortices in oblique angle, linked vortices
Collision Dynamics of Vortices

Commutative topological charge

- reconnection
- passing through

Non-commutative topological charge

- polar rung
- ferromagnetic rung
Collision Dynamics of Linked Vortices

<table>
<thead>
<tr>
<th>Commutative</th>
<th>Non-commutative</th>
</tr>
</thead>
<tbody>
<tr>
<td>untangle</td>
<td>not untangle</td>
</tr>
</tbody>
</table>
Algebraic Approach

Consider 4 closed paths encircling two vortices

Path $d$ defines vortex $B$ as $ABA^{-1}$ (same conjugacy class)
Collision of Vortices

topologically forbidden (only Abelian)
Linked Vortices

Linked vortices cannot untangle
Summary

1. Vortices with non-commutative topological charge are defined as non-Abelian vortices.
2. Non-Abelian vortices can be realized in the cyclic phase of spin-2 BEC.
3. Collision of two non-Abelian vortices create a new vortex between them as a rung (networking structure).
Future: Topological Charge of Linked Vortices

Do linked vortices themselves have a different topological charge from them as each vortices?
Future: Network Structure in Quantum Turbulence

Turbulence with Abelian vortices

• Cascade of vortices

Turbulence with non-Abelian vortices

• Large-scale networking structures among vortices with rungs
• Non-cascading turbulence

New turbulence!
Quantized Vortices in Multi-component BEC

Scalar BEC

$^4$He

$e^{i\phi}$

gauge

integer vortex

Polar in $S = 1$ BEC

$e^{i\phi} \cos \theta$

$\phi = 0$

$\phi = \pi$

gauge + headless vector

$^3$He-A

$\hat{d}(\hat{m} + i\hat{n})$

d vector + triad

1/2 vortex

$\pi$ gauge transformation

reverse of $d$ vector
### Quantized Vortices in Multi-component BEC

#### Abelian vortices

- **scalar BEC**
  - integer vortex

  \[
  \Psi \propto \exp[i\theta]
  \]

- **Polar phase in spin-1 spinor BEC**
  - \(\frac{1}{2}\) vortex

  \[
  \Psi \propto \frac{1}{\sqrt{2}} \begin{pmatrix} \exp[i\theta] \\ 0 \\ 1 \end{pmatrix}
  \]

#### non-Abelian vortices

- **Cyclic phase in spin-2 spinor BEC**
  - \(\frac{1}{2}\) spin vortex or \(\frac{1}{3}\) vortex

  \[
  \Psi \propto \frac{1}{2} \begin{pmatrix} i \exp[i\theta] \\ 0 \\ \sqrt{2} \\ 0 \\ i \exp[-i\theta] \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} \exp[i\theta] \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}
  \]

- We investigate detailed structure of vortices
Mean Field Approximation for BEC at $T = 0$

Spin-2

\[ H \simeq \int d\mathbf{x} \left[ -\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\text{tot}}^2 + \frac{c_1}{2} F^2 + \frac{c_2}{2} |A_{00}|^2 \right] \]

\[ c_0 = \frac{4g_2 + 3g_4}{7}, \quad c_1 = \frac{g_4 - g_2}{7}, \quad c_2 = \frac{7g_0 - 10g_2 + 3g_4}{7} \]

\[ n_{\text{tot}}(\mathbf{x}) = \Psi_m^*(\mathbf{x})\Psi_m(\mathbf{x}), \quad F(\mathbf{x}) = \Psi_m^*(\mathbf{x}) \hat{F}_{mm'}(\mathbf{x})\Psi_{m'}(\mathbf{x}) \]

\[ A_{00}(\mathbf{x}) = \frac{1}{\sqrt{5}} \left[ 2\Psi_2(\mathbf{x})\Psi_{-2}(\mathbf{x}) - 2\Psi_1(\mathbf{x})\Psi_{-1}(\mathbf{x}) + \Psi_0(\mathbf{x})^2 \right] \]

- $n_{\text{tot}}$: total density
- $F$: magnetization
- $A_{00}$: singlet pair amplitude
Spin-2 BEC

\[ H \simeq \int d\mathbf{x} \left[ -\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\text{tot}}^2 + \frac{c_1}{2} \mathbf{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right] \]

1. \( c_1 < 0 \rightarrow \text{ferromagnetic phase: } \mathbf{F} \neq 0 \)

2. \( c_1 > 0, \ c_2 < 0 \rightarrow \text{polar phase: } \mathbf{F} = 0, \ A_{00} \neq 0 \)

3. \( c_1 > 0, \ c_2 > 0 \rightarrow \text{cyclic phase: } \mathbf{F} = A_{00} = 0 \)

ferromagnetic: \( e^{i\phi} e^{-ie \cdot \hat{\mathbf{F}}_\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \)

polar: \( e^{i\phi} e^{-ie \cdot \hat{\mathbf{F}}_\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \)

or \( \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \)

or \( \begin{pmatrix} i/2 \\ 0 \\ 0 \\ i/2 \end{pmatrix} \)

cyclic: \( e^{i\phi} e^{-ie \cdot \hat{\mathbf{F}}_\alpha} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \)

or \( \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix} \)
Vortices in Spin-2 BEC

Wave function of each vortex

<table>
<thead>
<tr>
<th></th>
<th>gauge vortex</th>
<th>integer-spin vortex</th>
<th>1/2-spin vortex</th>
<th>1/3 vortex</th>
<th>2/3 vortex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{S} e^{i\theta}$</td>
<td>$\begin{pmatrix} i/2 &amp; 0 &amp; 0 \ 0 &amp; 1/\sqrt{2} &amp; 0 \ i/2 &amp; 0 &amp; 1/\sqrt{2} \end{pmatrix}$</td>
<td>$\begin{pmatrix} i \exp[2i\theta] &amp; 0 &amp; 0 \ 0 &amp; \sqrt{2} &amp; 0 \ i \exp[-2i\theta] &amp; 0 &amp; \sqrt{2} \end{pmatrix}$</td>
<td>$\begin{pmatrix} i \exp[i\theta] &amp; 0 \ 0 &amp; \sqrt{2} &amp; 0 \ i \exp[-i\theta] &amp; 0 \end{pmatrix}$</td>
<td></td>
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</tr>
</tbody>
</table>

$\hat{S}$ : arbitrary gauge transformation and spin rotation
Vortices in Spin-2 BEC

<table>
<thead>
<tr>
<th>vortices</th>
<th>gauge rotation</th>
<th>spin rotation</th>
<th>core structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>gauge</td>
<td>1</td>
<td>0</td>
<td>density core</td>
</tr>
<tr>
<td>integer spin</td>
<td>0</td>
<td>1</td>
<td>polar core</td>
</tr>
<tr>
<td>1/2 spin</td>
<td>0</td>
<td>1/2</td>
<td>polar core</td>
</tr>
<tr>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>ferromagnetic core</td>
</tr>
<tr>
<td>2/3</td>
<td>2/3</td>
<td>2/3</td>
<td>ferromagnetic core</td>
</tr>
</tbody>
</table>
Collision of Same Vortices

Energetically unfavorable

reconnection
Collision of Different Commutative Vortices

Energetically unfavorable

Passing
Collision of Different Non-commutative Vortices

Topologically forbidden

rung
Homotopy Group of the Cyclic Phase of Spin-2 BEC

Order parameter manifold: \( \frac{G}{H} \sim \frac{U(1)_G \times SO(3)_S}{T_{G+S}} \)

First homotopy group: \( \pi_1 \left( \frac{G}{H} \right) \sim \pi_1 \left( \frac{U(1)_G \times SO(3)_S}{T_{G+S}} \right) \)
\( \sim \pi_1 \left( \frac{U(1)_G \times SU(2)_S}{T^*_G+S} \right) \)
\( \frac{T^*}{Z_2} \sim T \)